NETAJI SUBHAS OPEN UNIVERSITY

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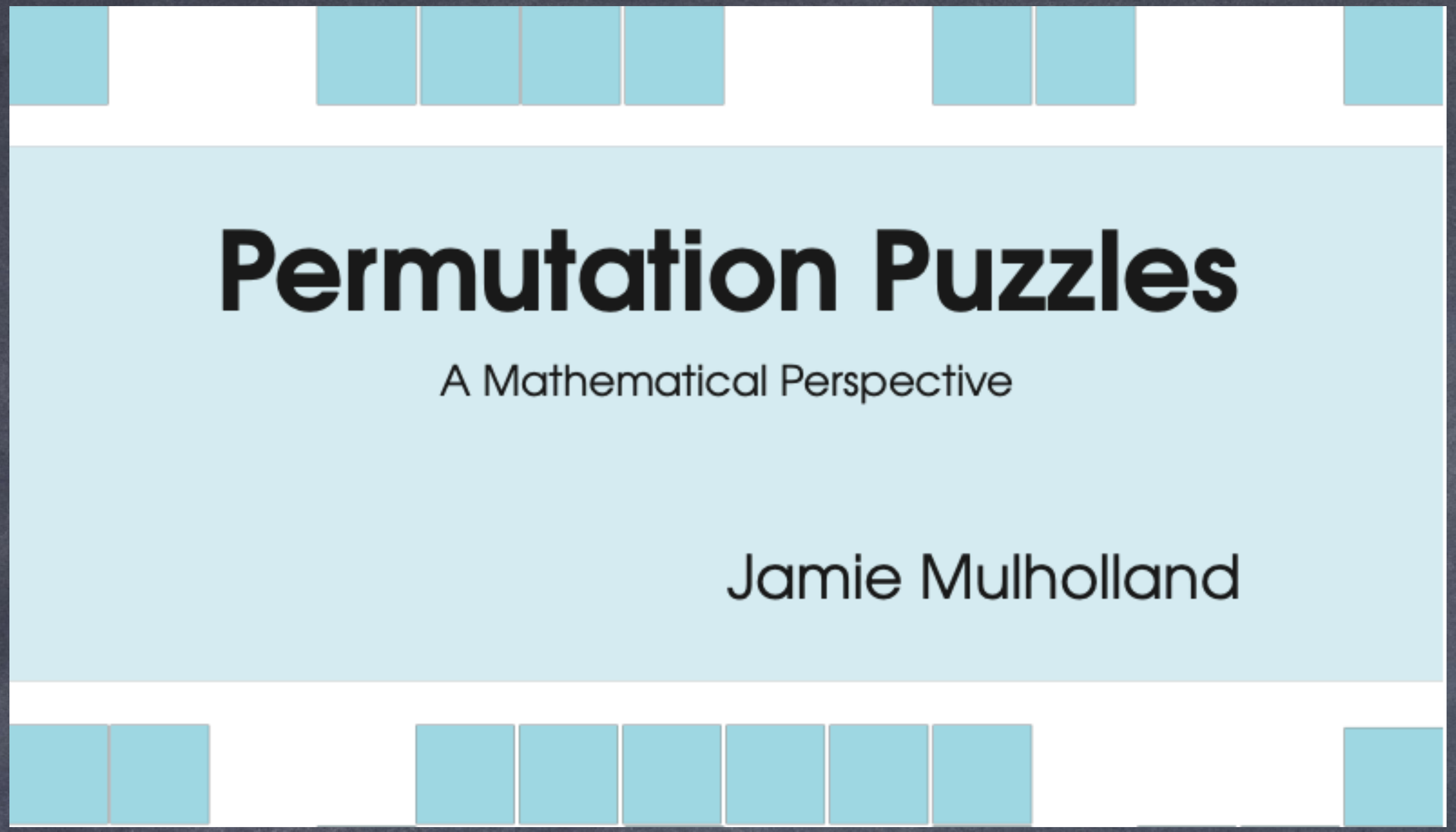
THE FIFTEEN PUZZLE

A game consisting of a 4 x 4 array, each carrying a tile numbered 1 to 15, the last slot is left blank and 14 is placed at the position (4,3) while 15 is placed at the position (4,2). In other words, the natural positions of 14 and 15 are swapped.



Legal Moves: Any tile whose neighbour is the blank slot can be pushed into the blank position.

Goal: To put the tiles back in the original order. Swap the position of 14 and 15 (simple, isn't it (?)!).

Associating a permutation with every state of the puzzle.

For the purposes of this discussion, we will think of a permutation σ simply as a self map of the set $\mathcal{S} := \{1,2,\ldots,15,\star\}$. The sequence $\sigma(i)$, $i \in \mathcal{S}$, is often identified with σ.

To turn the grid layout into a sequence, you could, for instance, line up the rows next to each other, in other words, read off the numbers from left-to-right and top-to-bottom:

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	⭐

1	2	3	4	5	6	7	8	9	10	11	12	13	15	14	⭐

(15 14)

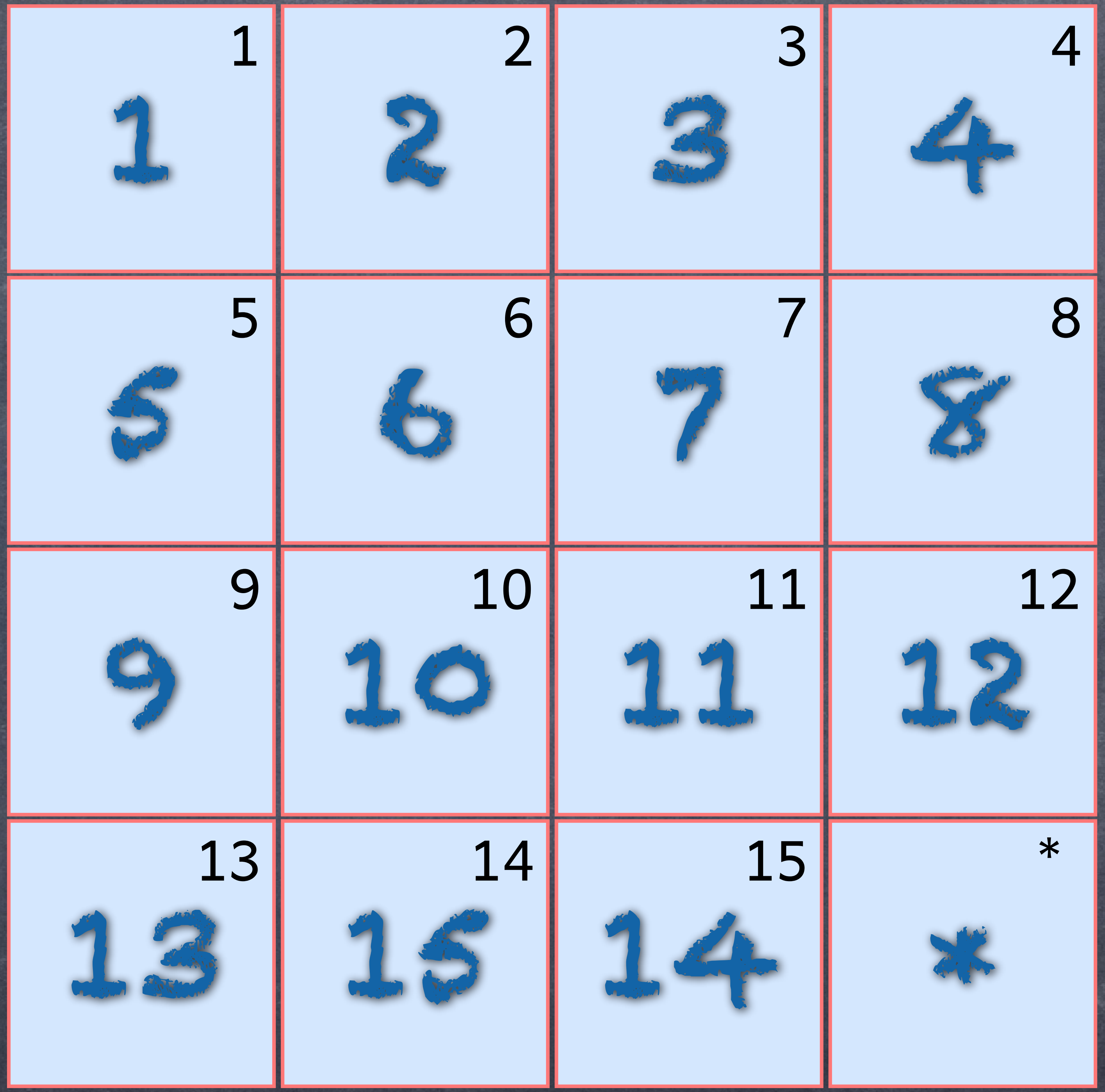
- A permutation is said to be a transposition if $\tau(i) = j$ and $\tau(j) = i$ for a pair $i \neq j$ and τ does not change any other value.

- Applying a transposition τ to an arbitrary permutation σ, we obtain new permutation:

$$(\tau \circ \sigma)(\ell) = \begin{cases} \sigma(j) & \text{if } \ell = i \\ \sigma(i) & \text{if } \ell = j \\ \sigma(\ell) & \text{otherwise} \end{cases}$$

- The identity permutation is special and is denoted by ι.

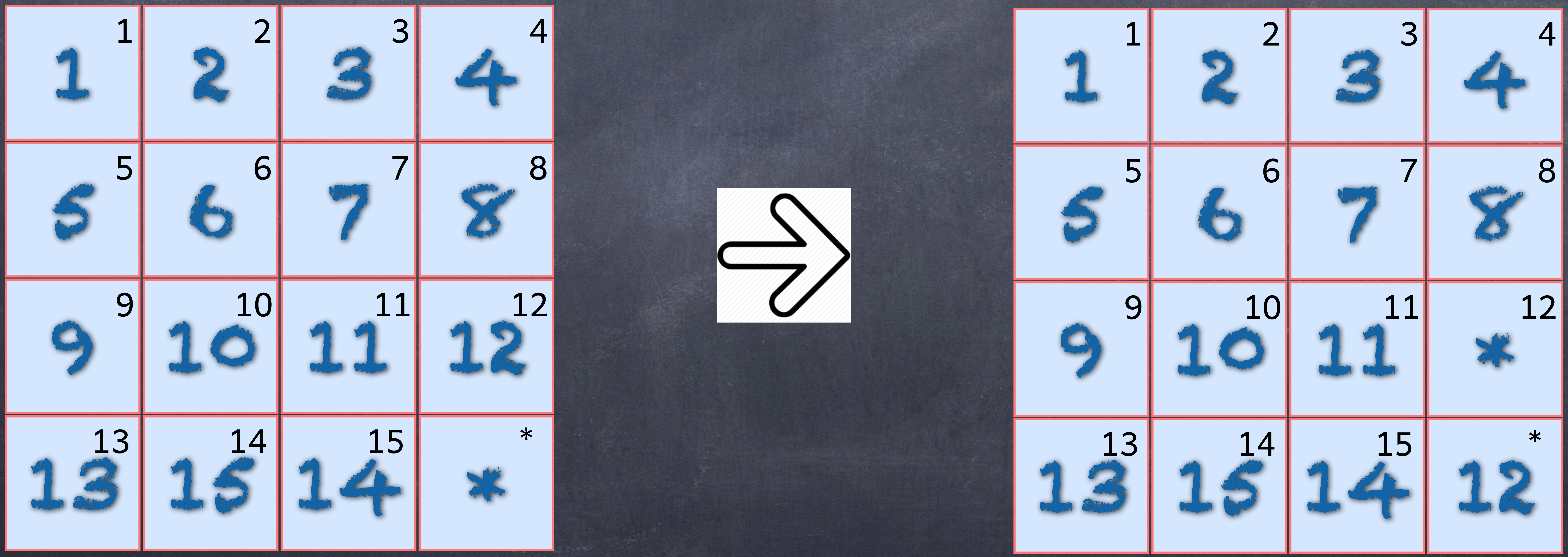
Every state of the game can be represented by a permutation of $\{1,2,\ldots,15\} \cup \{ \star \}$, where $\{ \star \}$ denotes the blank tile. The initial state



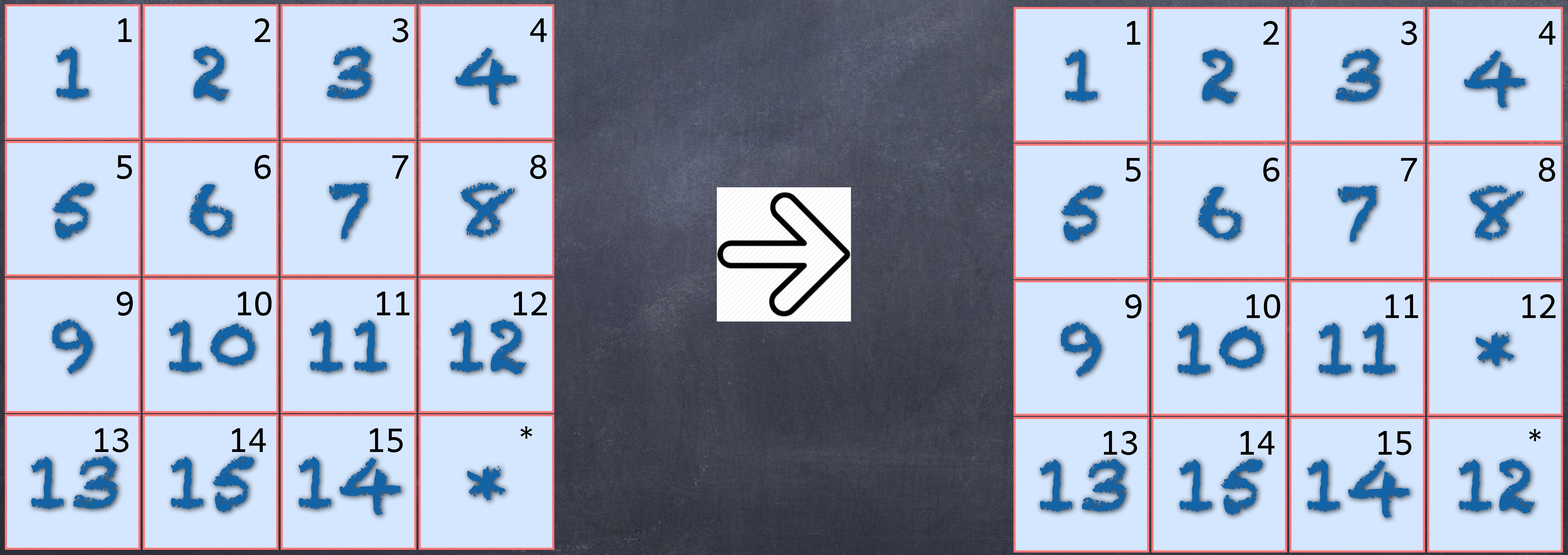
corresponds to the permutation given by the transposition

(15 14).

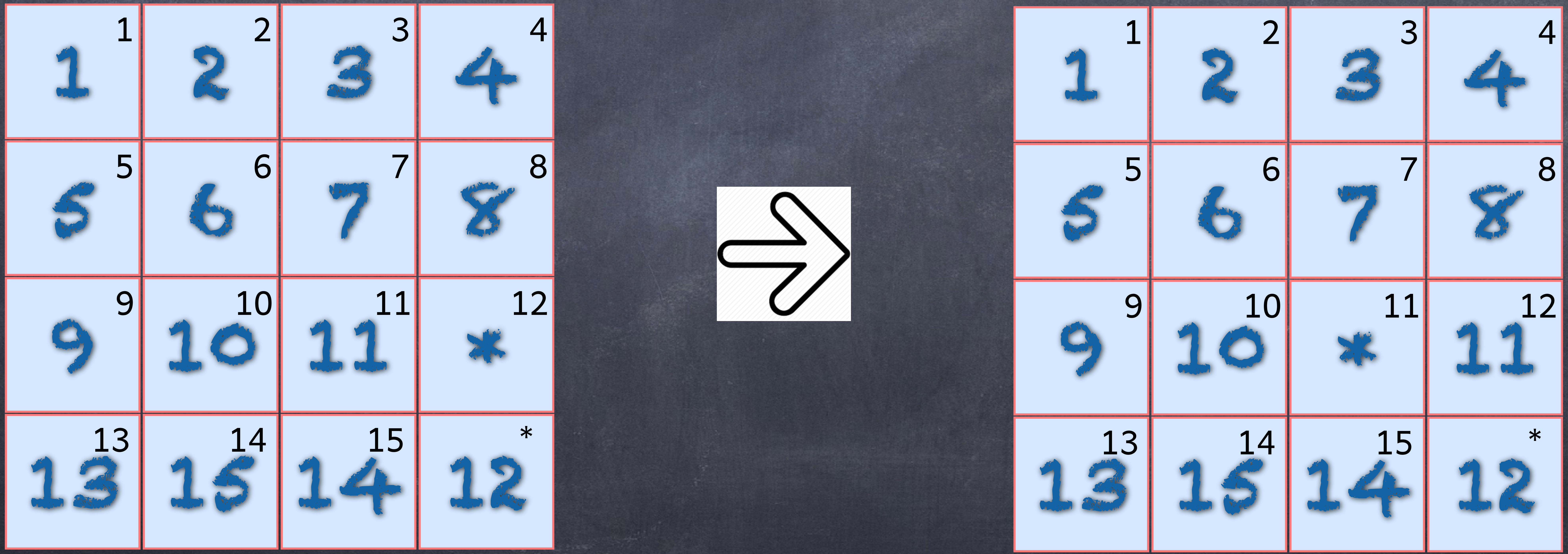
After the first legal move, the board would transform from the initial state to a new state. For example the new state shown below is represented by the product of two transpositions: (* 12) (15 14).



The transformation can also be represented by a permutation of the set $\mathcal{S}$.
In the example shown below, the transformation is represented by (* 12).



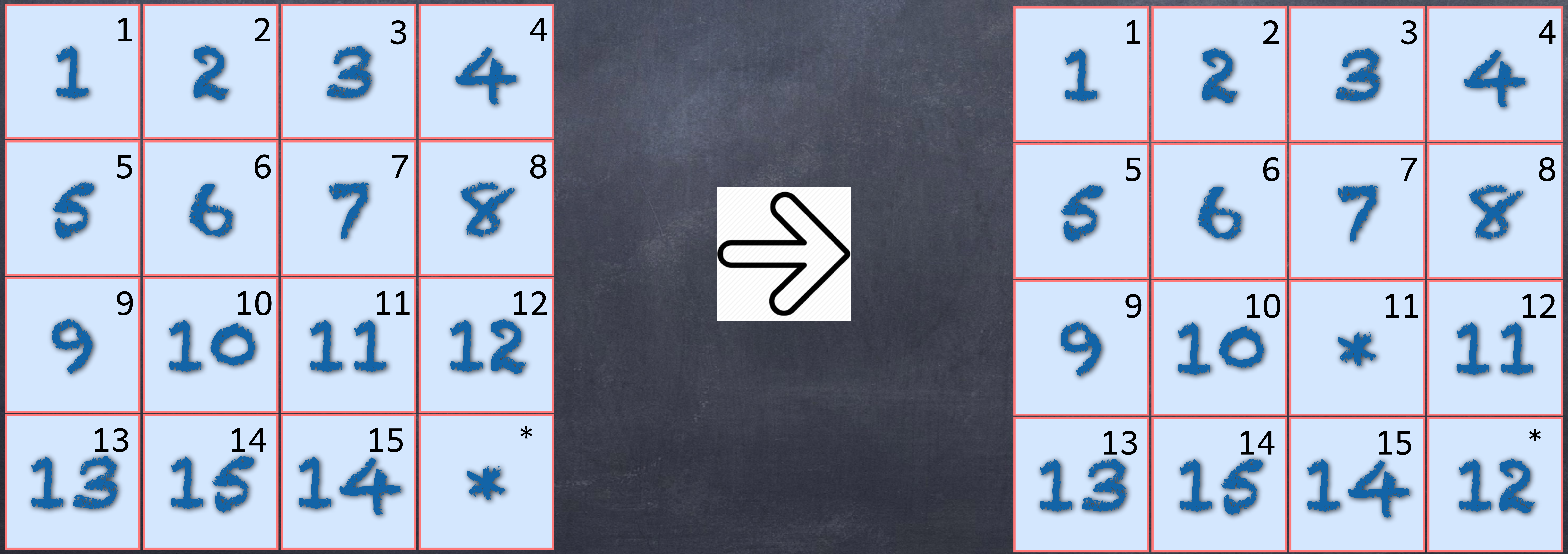
Following our first move by a second one, say (12 11), produces the new state shown below



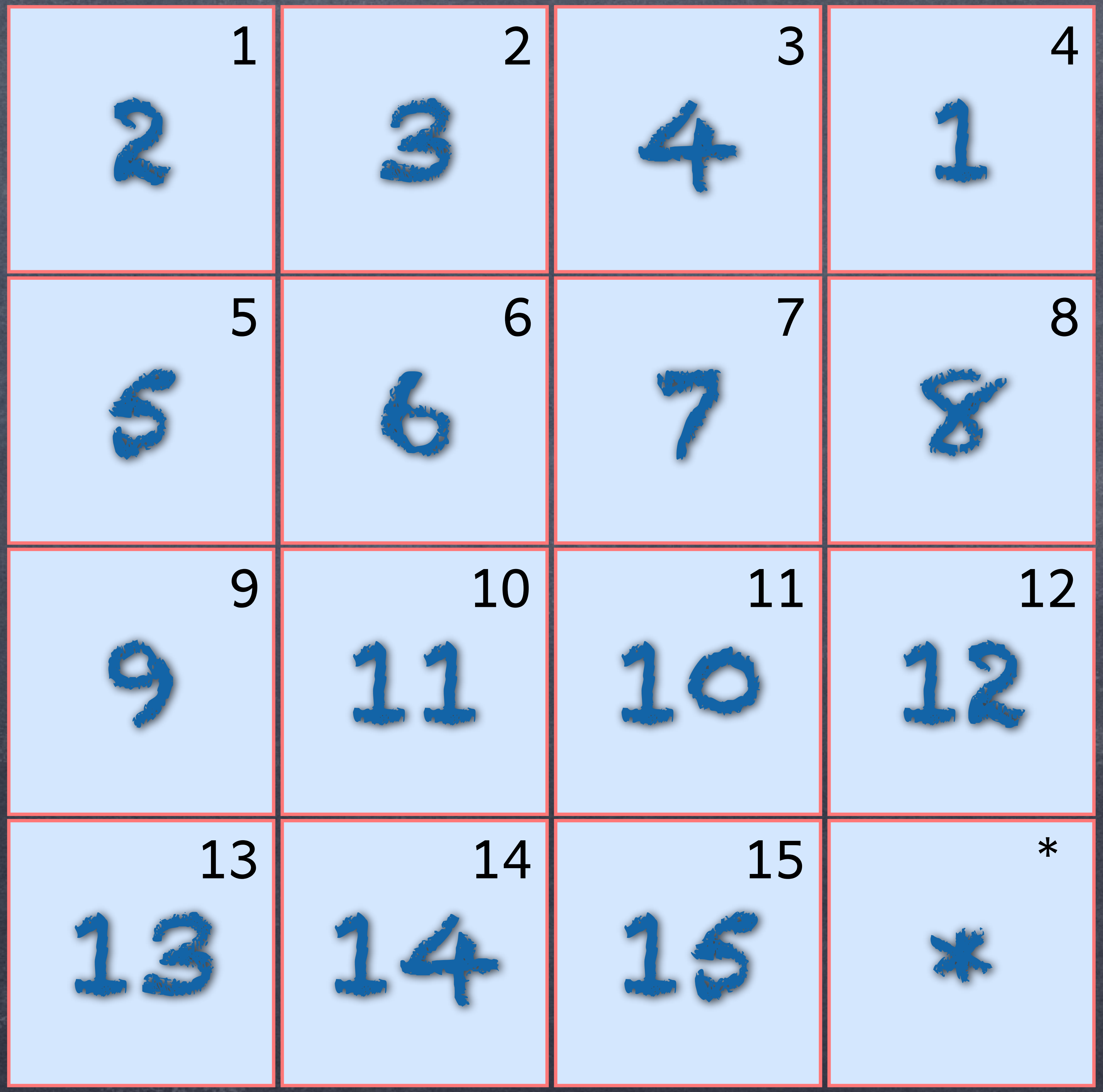
Thus after two legal moves (* 12) and (* 11), we have the initial and final state shown below. Recall that the initial state corresponds to the permutation (15 14). Now, observe that product (* 11) (* 12) (15 14) is the permutation,

$$(12\ 11)(\star\ 12)(15\ 14) = (11\ \ \star\ \ 12)(15\ 14)$$

Which is the permutation representing the final state.



Conversely, given any permutation, it determines a state of the game. For instance, if $\sigma = (1\ 2\ 3\ 4)(10\ 11)$, then it determines the state



Thus there is a one to one correspondence between states of the game and permutations. A legal move corresponds to applying a transposition to a state, or what is the same as a permutation. One move followed by another corresponds to applying transpositions, one after the other to a given a state to go to a new one.

The 15 puzzle therefore amounts to ascertaining if there exists a sequence of admissible transpositions corresponding to legal moves that will transform the initial permutation with 14 and 15 swapped to the identity permutation.

A permutation is a Product of transpositions

Every permutation σ can be obtained from the identity permutation by a sequence of transpositions. To prove this, start with the identity permutation, and repeat the following until the permutation at hand is the one you want to see:

Find a location i that's messed up in the current permutation, i.e, it doesn't have the element you need in there. Find where the element is in the current permutation, and if that's location j, you could perform a transposition between i and j. This fixes up the location i. In every step, you fix at least one location, and never mess up anything else: so at the end of at most n steps (assuming you are working with a sequence of n elements), you would be done.

For example, suppose the permutation you want to obtain is 3,4,2,1. Here is how the argument above would play out:

1. 1,2,3,4. The first location is messed up, so swap 1 and 3.
2. 3,2,1,4. The second location is messed up, so swap 2 and 4.
3. 3,4,1,2. The third location is messed up, so swap 1 and 2.
4. 3,4,2,1. Now we are done.

Note that this may not be the only way of performing a sequence of transpositions that can morph ι into σ — there may be various roads to σ. However, it turns out that no matter what route you take to transform $\iota \longrightarrow \sigma$, the number of steps you perform will always have the same parity.

The parity of a permutation

This partitions the set of all permutations into two categories:
even permutations: those permutations that are reachable from the identity with an even number of transpositions odd **permutations:** those permutations that are reachable from the identity with an odd number of transpositions.

The fact from the previous slides above makes this classification unambiguous.

We begin with the observation that every move in the game is really a transposition behind the scenes.

The impossibility proof

The permutation corresponding to our target state is the following:

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	⭐

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	⭐

The proof

In terms of the state of the game, notice that the final state has the blank tile at the bottom-right corner, just like we had at the start state. This means that in a hypothetical sequence of moves that morphs the initial game state into this solved state, we must have performed:

- An equal number of left and right moves; and
- An equal number of up and down moves.

If this is not the case — imagine the blank tile traveling through the board as you perform the moves — if the number of times you moved in opposite directions did not exactly cancel, it would be impossible for the blank location to be back at it's original location.

So in any winning sequence, the number of moves performed must be even. This implies that the permutation corresponding to the start state, in particular, can be obtained from the identity permutation with an even number of transformations. That makes the starting permutation an even permutation.

Since the empty space moves around the puzzle and then eventually returns home, the number of moves must be even.

1 1	2 2	3 3	4 4
5 5	6 6	7 7	8 8
9 9	10 10	11 11	12 12
13 13	14 14	15 15	16 *

the empty space must start in shaded box 16, and after each move it alternates the colour of the box it is in, and so if it returns to a shaded box it must have moved an even number of times.

This shows that every solvable state that places a blank tile at the bottom-right corner must correspond to an even permutation. This does not automatically imply that all such states associated with even permutations are solvable — it just shows that states with blank tiles at the bottom-right corner corresponding to odd permutations are firmly out of reach.

However, the permutation corresponding to the start state that we have been handed out is clearly an odd permutation: it can be obtained from the identtiy permutation by a transposition of the elements at the 14-th and 15-th positions.

So, we have established the impossibility of solving the 15 puzzle.

The main theorem

Continuing, on the theme from the previous slide, we ask what initial states can be solved? The answer:

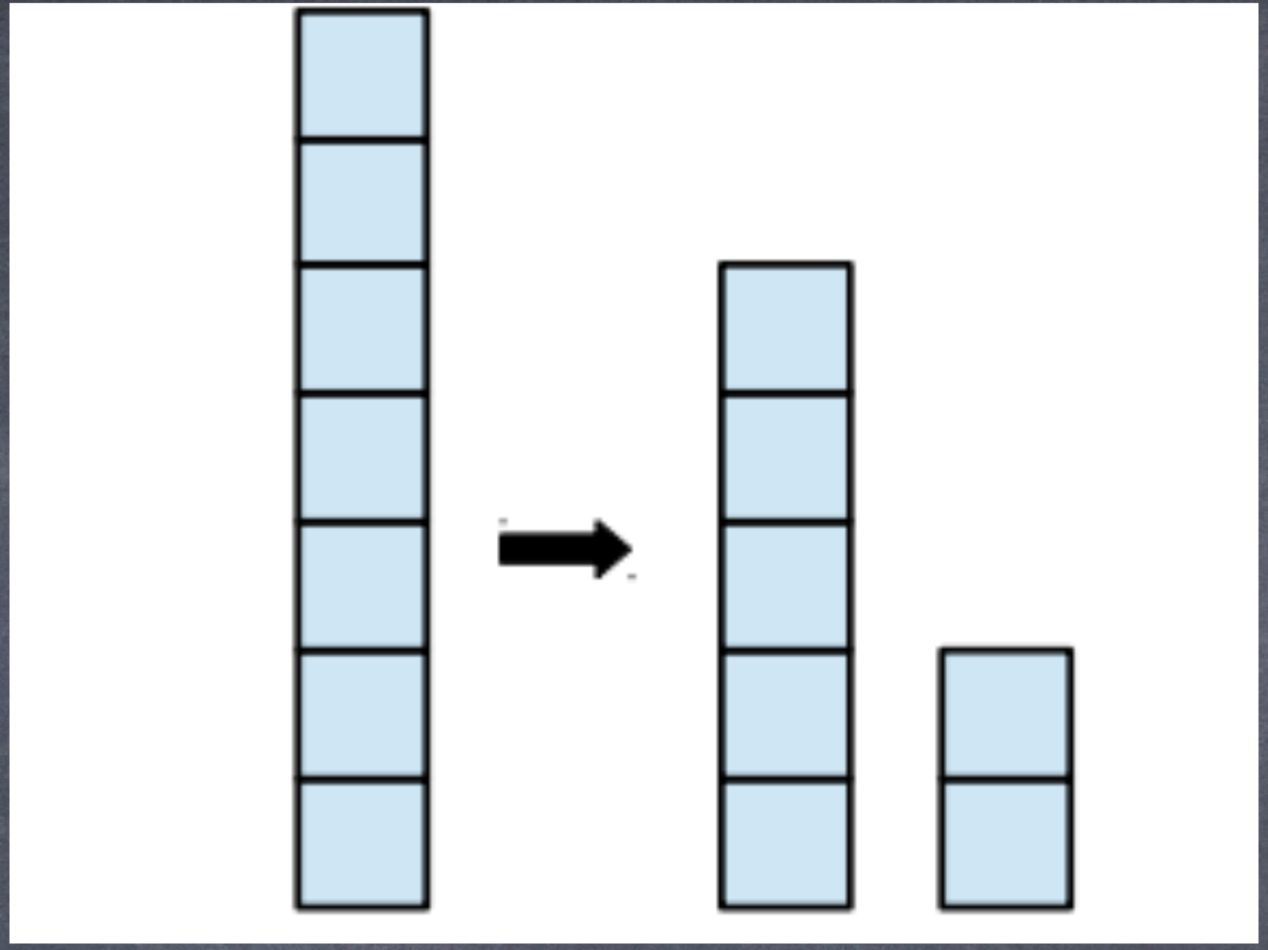
Theorem: A permutation σ of the 15-puzzle which fixes $\star$, is solvable if and only if it is even: i.e., $\sigma \in A_{15}$.

There are two directions we need to prove: A solvable configuration is an even permutation, and every even permutation is a solvable configuration. We have seen the proof of the first direction.
The second depends on the fact that even permutations can be expressed as products of 3-cycles.

THE UNSTACKING GAME

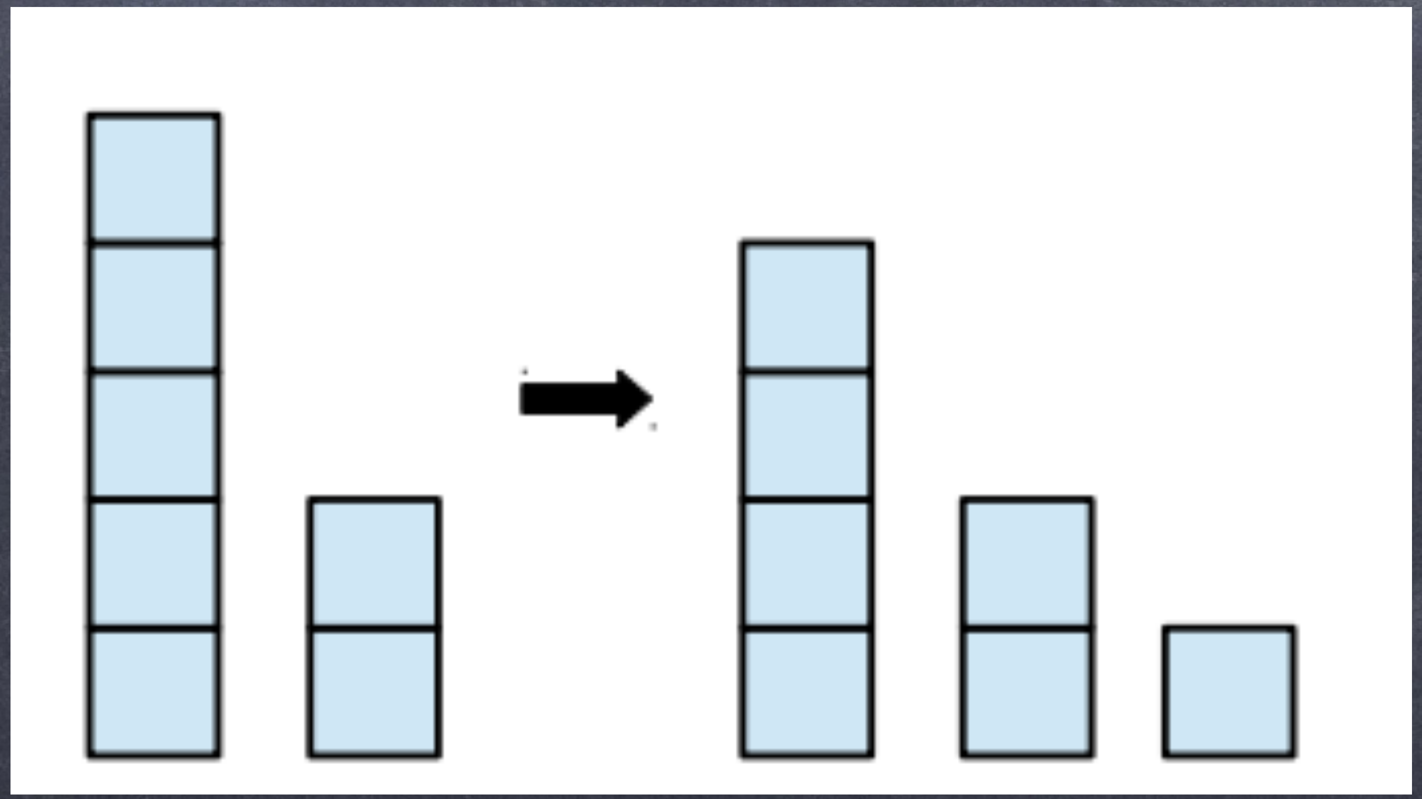
Given $(n+1)$ bricks stacked one on top of the other, split them into a set of two piles, one of them consisting of n_1 bricks and the other n_2 bricks such that $n_1 + n_2 = n + 1$. Suppose $n = 6$, then this first step may look like this:

You get $5 \times 2 = 10$ points for this first step



Now, you continue the game by splitting one of the two piles again, say the one with 5 bricks into two piles of 4 bricks and 1 brick:

This time, you earn $4 \times 1 = 4$ points.



The game ends when each pile has only one brick left.

The number of points you earn is the total of the points you have earned at each step.

Goal:

Given a n, find a strategy to maximise the number of points earned.

What follows is an example with $n = 7$.

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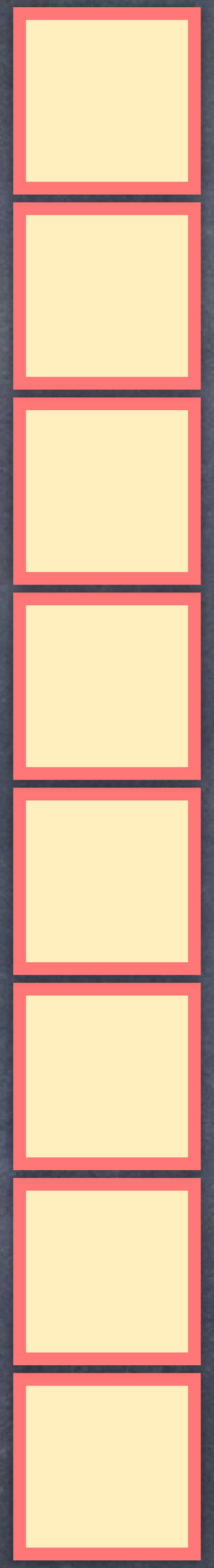
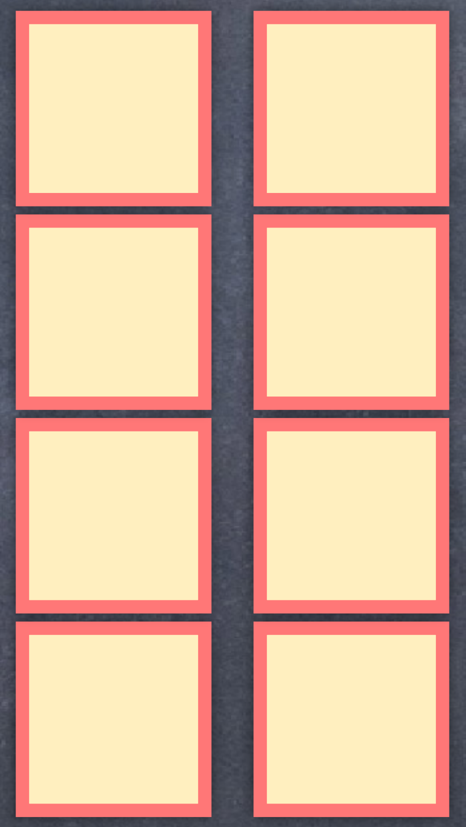


SCORE

16



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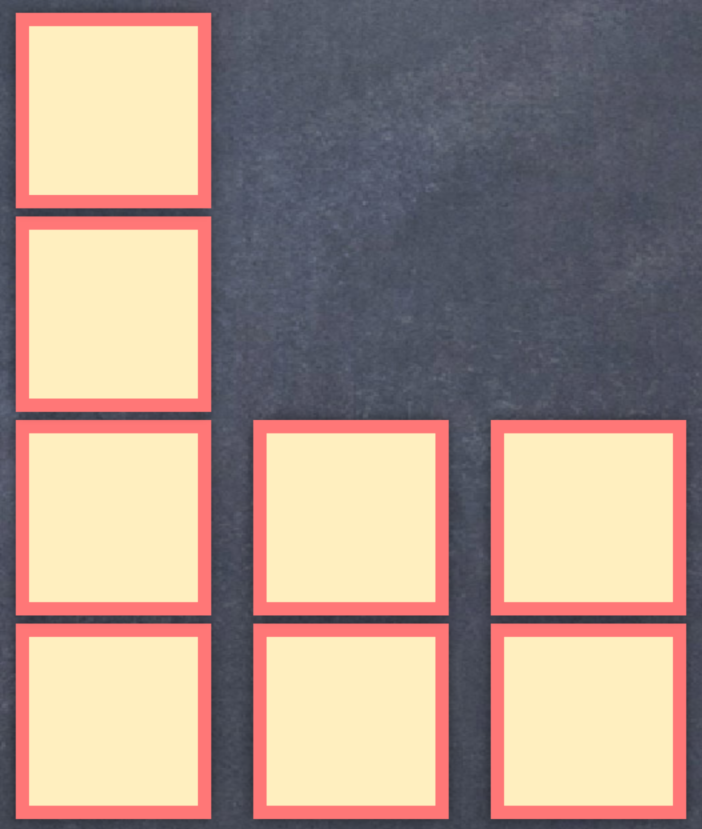
16 + 4



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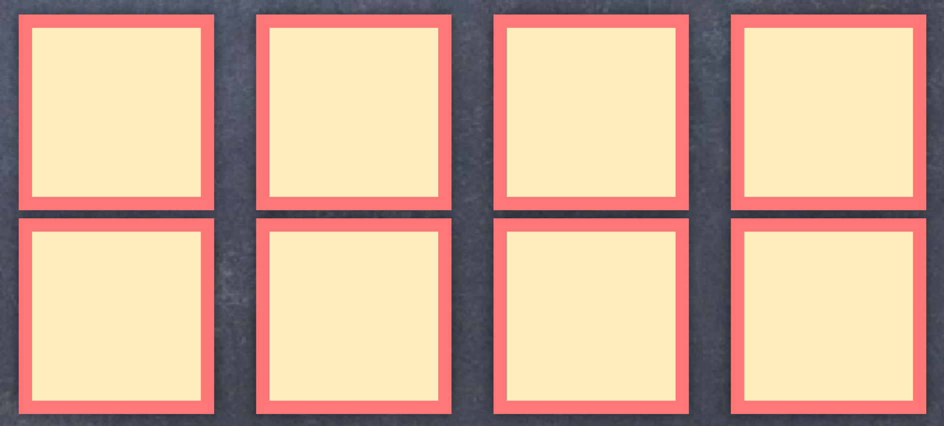
16 + 4 + 4



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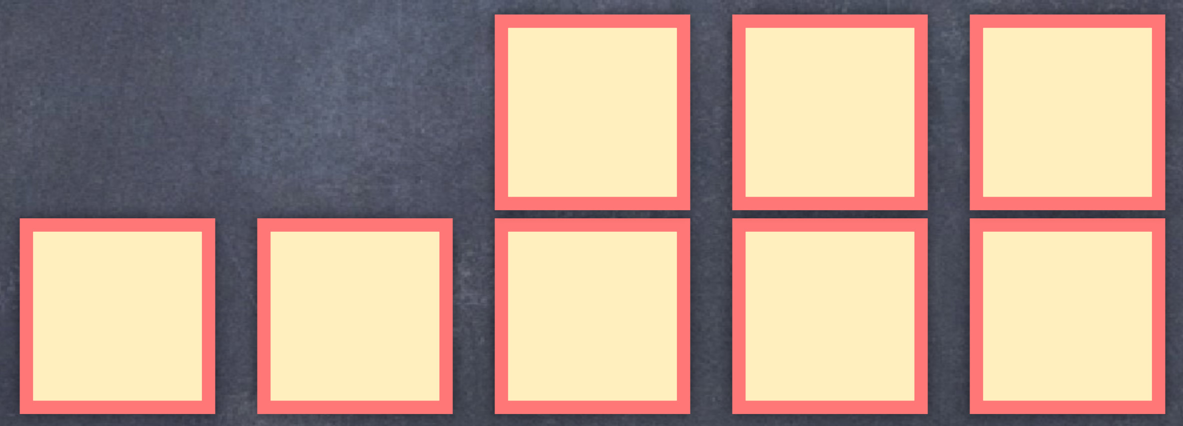
16 + 4 + 4 + 1



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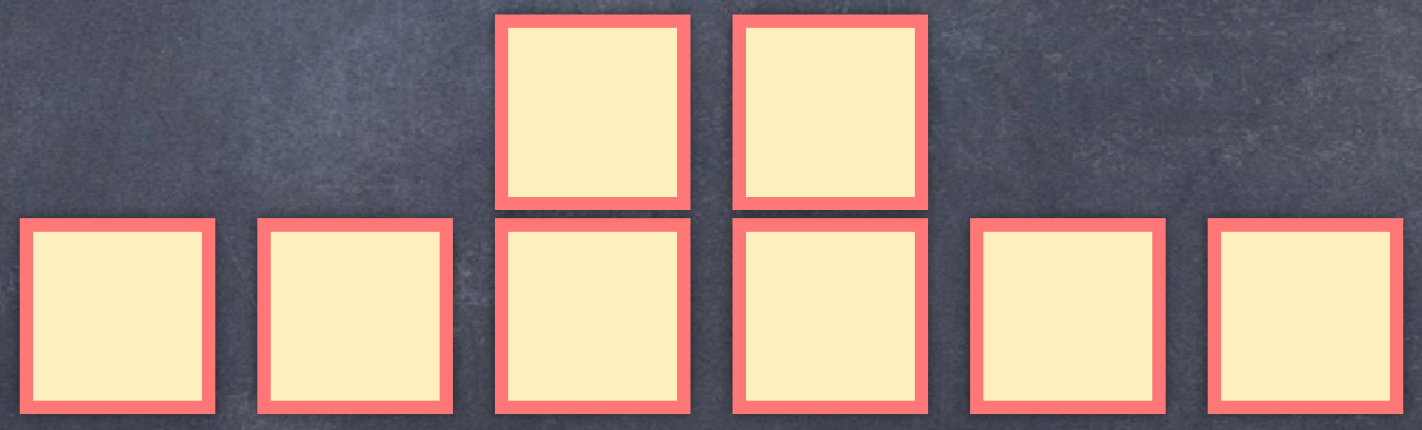
16 + 4 + 4 + 1 + 1



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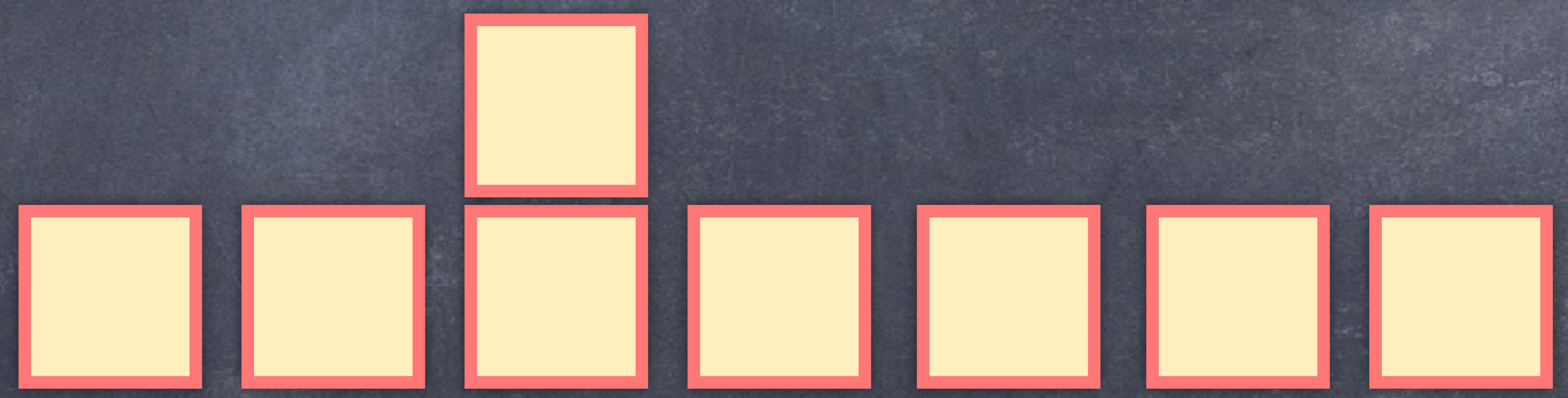
$16 + 4 + 4 + 1 + 1 + 1$



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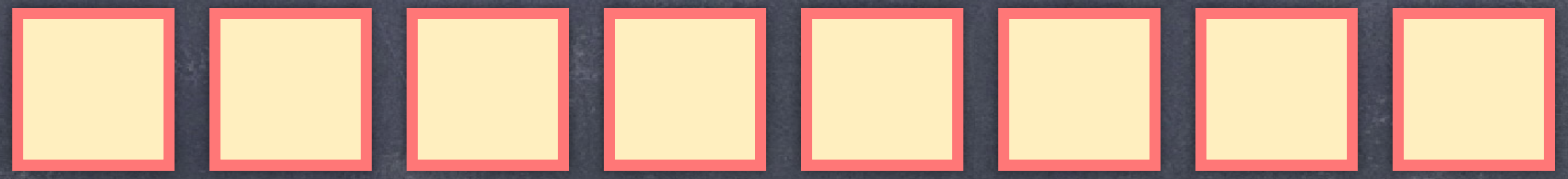
$16 + 4 + 4 + 1 + 1 + 1 + 1$



SCORE

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Big Question: Can you do better? Can you do any worse?

A different way you could play the game with any n is the following.

$$n+1 \to 1, n, \qquad \text{Points earned: } n$$

$$n \to 1, n-1, \qquad \text{Points earned: } n-1$$

$$n-1 \to 1, n-2, \qquad \text{Points earned: } n-2$$

$$\vdots \qquad\qquad \vdots$$

Continuing in this manner, after n steps, we will have nothing to split.

The points we would have earned in the bargain is $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$

Theorem: No matter what strategy is used, the score for the unstacking game with $n+1$ blocks is $\frac{n(n+1)}{2}$.

The proof is by the principle of strong induction:

Let $P(n)$ be a property that applies to natural numbers. Suppose that the following are true:

$P(0)$ is true. For any k, if $k \in \mathbb{N}$, $P(0), P(1), \ldots, P(k)$ are true, then $P(k+1)$ is true.

Then for any $n \in \mathbb{N}$, $P(n)$ is true.

The proof

For our base case, we prove $P(0)$, that any strategy for the unstacking game with one block will always yield $\frac{0(0+1)}{2} = 0$ points.

This is true because the game immediately ends if the only stack has size one, so all strategies immediately yield 0 points.

For the inductive hypothesis, assume that for some $n \in \mathbb{N}$ and all $k \in \mathbb{N}, k \leq n$, $P(k)$ holds.

Under this hypothesis, to show that P(n+1) holds.

Finishing the proof

Since each stack must have at least one block in it, this means that $k \geq 0$ (so that $k + 1 \geq 1$) and that $k \leq n$ (so that $(n - k) + 1 \geq 1$).

Consequently, we know that $0 \leq k \leq n$, and by the inductive hypothesis we have that the total number of points earned from splitting the stack of $(k+1)$ blocks down must be $\frac{k(k+1)}{2}$.

Similarly, since $0 \leq n - k \leq n$, again by the inductive hypothesis, the total score for the stack of $(n-k)+1$ blocks must be $\frac{(n-k)(n-k+1)}{2}$.

The last step in the proof

Let us consider the total score for this game. The initial move yields $(k+1)(n-k+1)$ points.

The two subgames yield $\frac{k(k+1)}{2}$ and $\frac{(n-k+1)(n-k)}{2}$ points, respectively.

This means that the total number of points earned is

$$(k+1)(n-k+1) + \frac{(n-k+1)(n-k)}{2} + \frac{k(k+1)}{2} = \frac{(n+1)(n+2)}{2}.$$

This is the answer with a stack of size $n+2$. The inductive step is therefore verified, completing the proof.

THANKS!